9 Sequences of Functions

9.1 Pointwise and uniform convergence

Let $D \subseteq \mathbb{R}^d$, and let $f_n : D \to \mathbb{R}$ be functions $(n \in \mathbb{N})$.

We may think of the functions $f_1, f_2, f_3, \ldots$ as forming a **sequence of functions**.

(f_n) or $(f_n)_{n=1}^{\infty}$

This is very different from a sequence of numbers but it is still possible to define the concept of a limit of such a sequence.

Such a notion is quite important.

For instance when solving differential equations or other problems one is often able to produce a sequence of approximate solutions and then needs to know in which sense the approximate solutions converges to the exact one.

We shall discuss two of the main notions of convergence of sequences of functions $f_n : D \to \mathbb{R}$.

To help us, recall the following **non-standard terminology**, introduced earlier.

Absorption

Definition 9.1.1 Let $(\boldsymbol{x}_n) \subseteq \mathbb{R}^d$ and let $A \subseteq \mathbb{R}^d$.
We say that the set A **absorbs** the sequence $(\boldsymbol{x}_n)$ if there exists $N \in \mathbb{N}$ such that the following condition holds:

$$\text{for all } n \geq N, \text{ we have } \boldsymbol{x}_n \in A, \qquad (*)$$

i.e., all terms of the sequence from $\boldsymbol{x}_N$ onwards lie in the set A.

The following is a slight variation of an exercise on a question sheet.

The proof is an **exercise**.

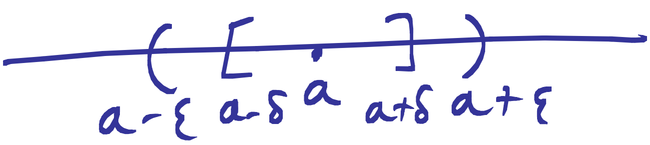
Proposition 9.1.2 Let $a \in \mathbb{R}$ and let $(x_n) \subseteq \mathbb{R}$. Then the following statements are equivalent:

(a) the sequence (x_n) converges to a;

(b) for all $\varepsilon > 0$, the closed interval $\underbrace{[a - \varepsilon, a + \varepsilon]}$ absorbs the sequence (x_n).

standard version uses open intervals.

Say $\delta = \varepsilon/2$.



Can quote this result as standard now.

$f_n : D \to \mathbb{R}$, $f : D \to \mathbb{R}$ real-valued functions

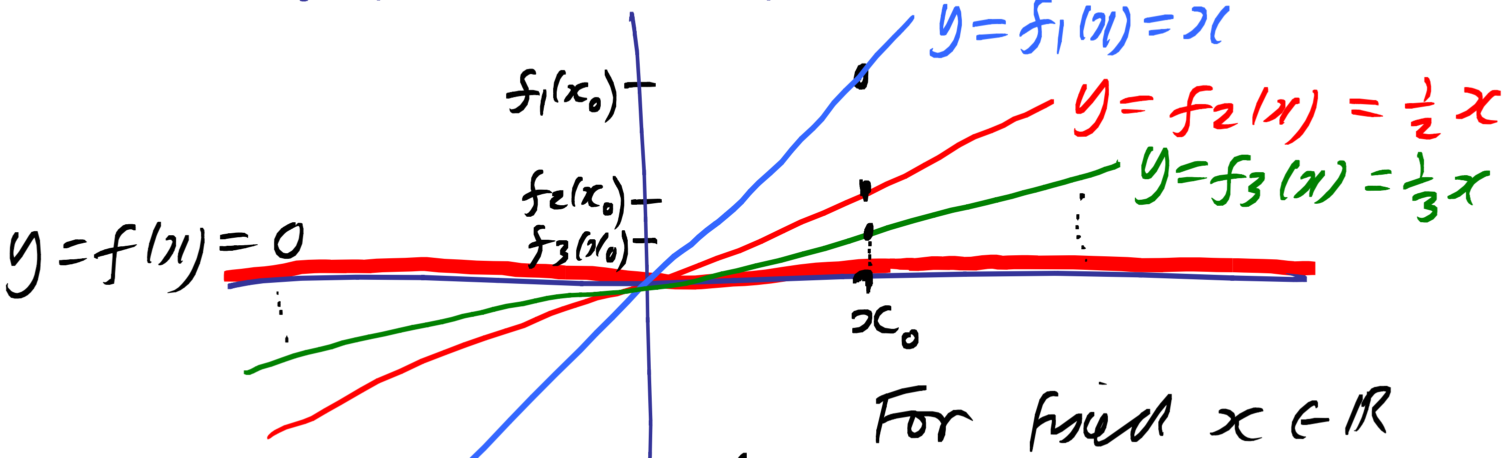
Definition 9.1.3 The sequence (f_n) **converges pointwise (on D)** to the function f if, for every $\boldsymbol{x} \in D$, the sequence $\underbrace{(f_n(\boldsymbol{x}))_{n=1}^{\infty}}_{\subseteq \mathbb{R}}$ converges to $\underset{\in \mathbb{R}}{f(\boldsymbol{x})}$, i.e., $f_n(\boldsymbol{x}) \to f(\boldsymbol{x})$ as $n \to \infty$.

Gap to fill in

Example.

Consider $f_n : \mathbb{R} \to \mathbb{R}$

$$f_n(x) = x/n. \qquad \begin{pmatrix} n \in \mathbb{N} \\ x \in \mathbb{R} \end{pmatrix}$$



For fixed $x \in \mathbb{R}$

$f_{1000}(1000) = 1$ etc.

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{x}{n} = 0.$$

The notion of uniform convergence is more subtle.

To explain this, we first extend our notions of closed ball and of sets absorbing sequences.

We need to consider sets and sequences of **functions**.

Definition 9.1.4 Let D be a non-empty subset of $\mathbb{R}^d$, let $f : D \mapsto \mathbb{R}$, and let (f_n) be a sequence of functions from D to $\mathbb{R}$.

For $\varepsilon > 0$, we define the **closed ball** centred on f and with radius ε, $\bar{\boldsymbol{B}}_\varepsilon(f)$, by ε = "radius" of ball.

$$\bar{\boldsymbol{B}}_\varepsilon(f) = \{g : D \to \mathbb{R} \mid |g(x) - f(x)| \le \varepsilon \text{ for all } \boldsymbol{x} \in D\}$$

$$= \{g : D \to \mathbb{R} \mid g(\boldsymbol{x}) \in [f(\boldsymbol{x}) - \varepsilon, f(\boldsymbol{x}) + \varepsilon] \text{ for all } \boldsymbol{x} \in D\}.$$

Note that this closed ball is a set of **functions** from D to $\mathbb{R}$.

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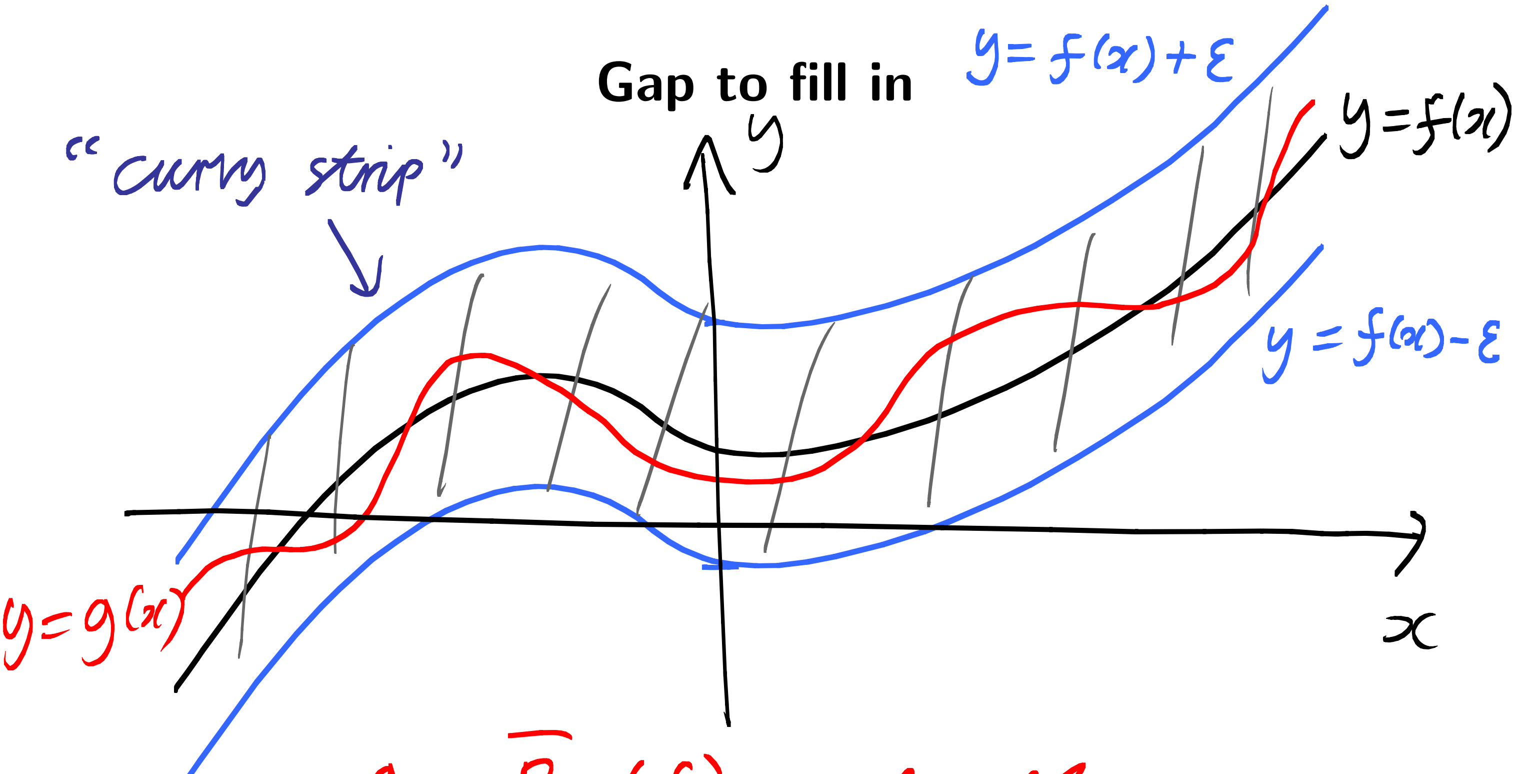


$g \in \bar{B}_\varepsilon(f)$ means

$f(x) - \varepsilon \le g(x) \le f(x) + \varepsilon$ for all x in the domain D (for this sketch we are thinking of $D = \mathbb{R}$)

As in $(*)$ above, we say that the closed ball $\bar{\boldsymbol{B}}_\varepsilon(f)$ **absorbs** the sequence of functions (f_n) if there exists $N \in \mathbb{N}$ such that the following condition holds:

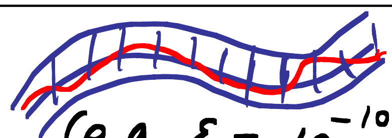
$$\text{for all } n \geq N, \text{ we have } f_n \in \bar{\boldsymbol{B}}_\varepsilon(f), \qquad (**)$$

i.e., all terms of the sequence of functions from f_N onwards lie in $\bar{\boldsymbol{B}}_\varepsilon(f)$.

The sequence (f_n) **converges uniformly (on D)** to the function f if, for every $\varepsilon > 0$, the closed ball $\bar{\boldsymbol{B}}_\varepsilon(f)$ absorbs the sequence (f_n).

In full, this means the following:

Strip could be very thin



(e.g. $\varepsilon = 10^{-100}$)

For all $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that, for all $n \geq N$ and all $\boldsymbol{x} \in D$, we have

$$|f_n(x) - f(x)| \leq \varepsilon\,.$$

The N in the full definition of uniform convergence depends only on ε; the same N works for all $\boldsymbol{x} \in D$.

Roughly this means that the sequences of numbers $(f_n(\boldsymbol{x}))$ converge to $f(\boldsymbol{x})$ at the same rate.

Pointwise convergence on the other hand means simply that the sequence of numbers $(f_n(\boldsymbol{x}))$ converges to $f(\boldsymbol{x})$ for each $\boldsymbol{x} \in D$.

At different points the speed of convergence could be very different.

The next result follows directly from the definitions.

(**Exercise.** Convince yourself that this is correct.)

Lemma 9.1.5 If (f_n) converges uniformly to f (on D) then it converges pointwise to f on D.

Uniform convergence $\Rightarrow$ pointwise convergence.
$\not\Leftarrow$

The converse of this lemma is NOT true.

It is time for some examples to illustrate this.

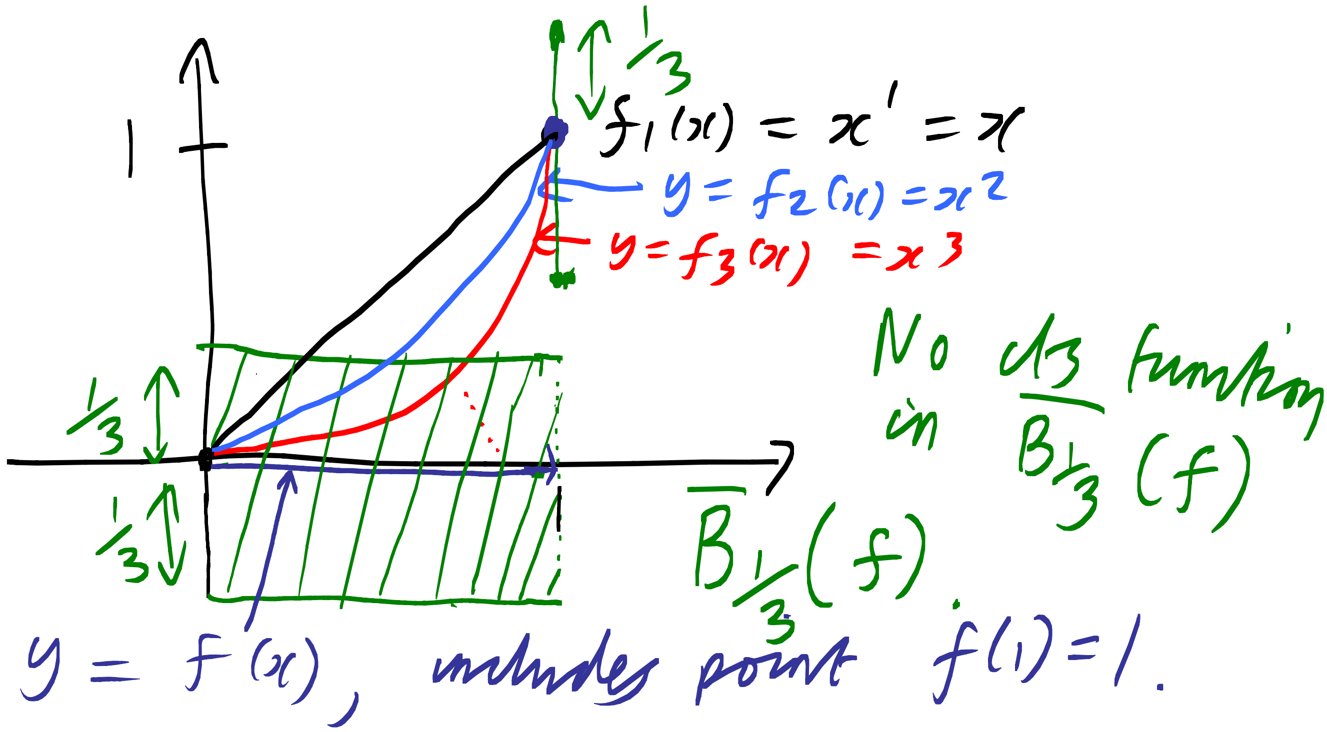
Ex. Look at earlier example

$f_n(x) = \frac{1}{n}x$. Show convergence is NOT uniform on $\mathbb{R}$.

Examples 1) Let $D = [0,1]$ and $f_n(x) = x^n$. Then (f_n) converges pointwise, but NOT uniformly, to the function

$$f(x) = \begin{cases} 0 & \text{if } 0 \le x < 1 \\ 1 & \text{if } x = 1. \end{cases}$$

Gap to fill in



Check pointwise convergence: Case by Case analysis.

<u>Case I.</u> When $x = 1$, then

$f_n(x) = x^n = 1$ for all n, so we have $\lim_{n\to\infty} f_n(1) = 1$.

Case II. Otherwise, consider x with $0 \le x < 1$.

Fix x, look at sequence

$$f_n(x) = x^n.$$

Since $|x| < 1$, we know $\lim_{n \to \infty} x^n = 0$.

So, in this case, $\lim_{n \to \infty} f_n(x) = 0$.

Failure of uniform convergence.

Uniform convergence: for every $\varepsilon > 0$, $\overline{B}_\varepsilon(f)$ must absorb the sequence (f_n).

Just need one "bad" ε for uniform convergence to fail.

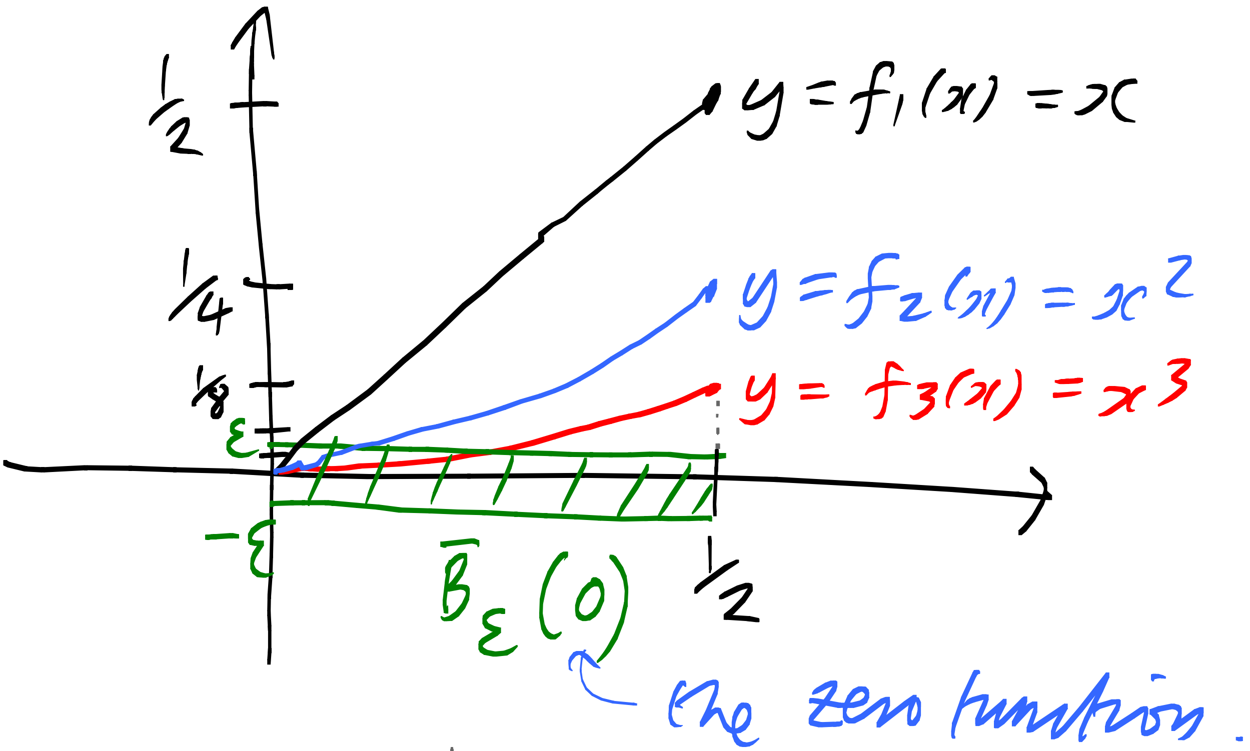
Above, with $\varepsilon = \frac{1}{3}$,

$\overline{B}_\varepsilon(f)$ does not absorb (f_n).

So the sequence does not converge uniformly to f.

2) Suppose $D = [0, 1/2]$ and again $f_n(x) = x^n$. Now (f_n) converges uniformly to the function which is identically 0 i.e. the function given by $f(x) = 0$ for all $x \in [0, 1/2]$ (we say it converges uniformly to 0).

Gap to fill in



$0 \leq f_1(x) \leq \frac{1}{2}$ $\quad x \in [0, \frac{1}{2}]$

$0 \leq f_2(x) \leq \frac{1}{4}$

$\vdots$

$0 \leq f_n(x) \leq 2^{-n}$ $\quad x \in [0, \frac{1}{2}]$.

Let $\varepsilon > 0$. Then, choose $n_0 \in \mathbb{N}$ such that $2^{-n_0} < \varepsilon$.

Then, for $n \geq n_0$, we have, for all x in $[0, \frac{1}{2}]$,

$$|f_n(x) - 0| \leq 2^{-n} \leq 2^{-n_0} < \varepsilon.$$

Thus, $\boxed{\text{for } n \geq n_0,\ f_n \in \overline{B}_\varepsilon(0)}$.

This shows that $f_n \to 0$ uniformly on $[0, \frac{1}{2}]$.

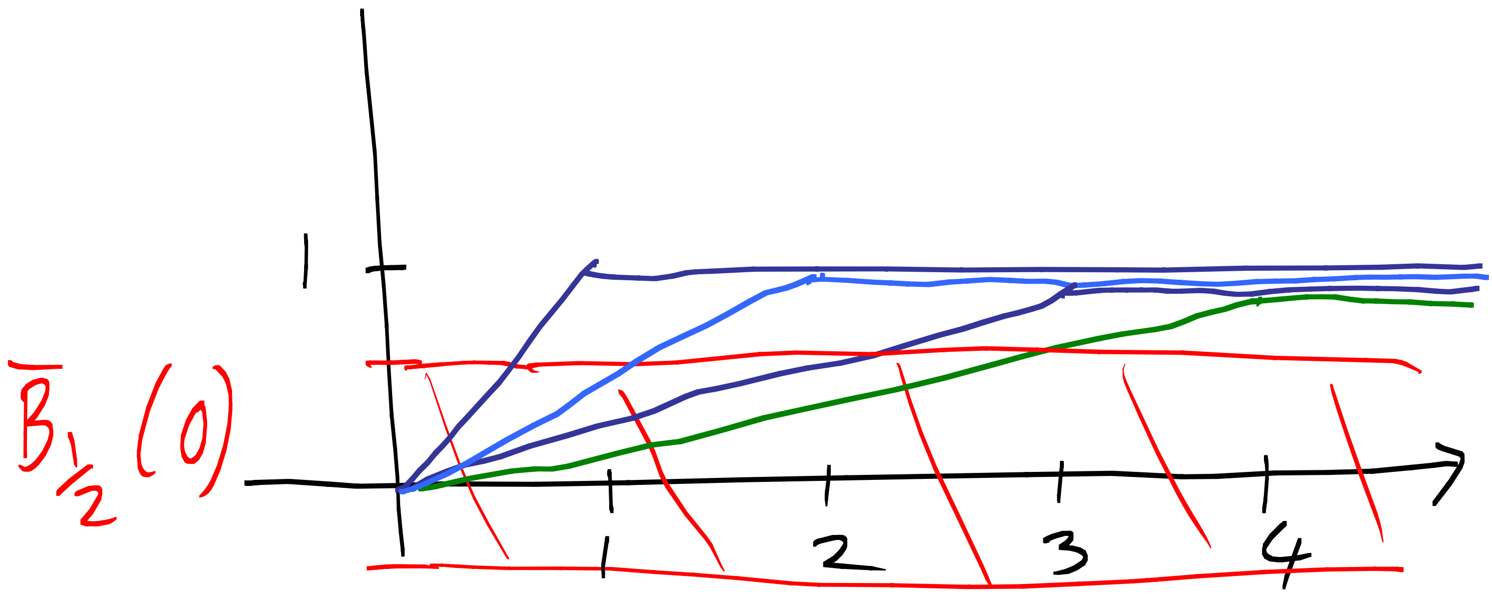
Note here that n_0 depends on ε but <u>NOT on x</u>.

3) Suppose $D = \mathbb{R}_+$ and $f_n : \mathbb{R}_+ \to \mathbb{R}$ is defined by

$$f_n(x) = \begin{cases} x/n & \text{if } 0 \le x \le n \\ 1 & \text{if } x > n. \end{cases}$$

Then the sequence of functions (f_n) converges to 0 pointwise, but not uniformly, on $\mathbb{R}_+$.

Gap to fill in



$\overline{B}_{\frac{1}{2}}(0)$ does not absorb the sequence (f_n). So sequence does not converge uniformly to 0.

Exercise: check pointwise convergence.

We will discuss a variety of methods to investigate the convergence of sequences of functions.

For an alternative approach, involving the **uniform norm**, see question sheets.

The above examples indicate that uniform convergence is much stronger than pointwise convergence and more difficult to establish.

One of the reasons that uniform convergence is important is that it has much better properties.

In the above examples, all of the functions f_n are continuous.

Unfortunately, as Example 1), shows the limit of a pointwise convergent sequence of continuous functions need not be continuous.

For uniformly convergent sequences the situation is much better.

The **proof** of the next theorem is **NEB** (not examinable as bookwork): see Wade's book, Theorem 7.9, if you are interested.

The **statement and applications of this theorem are examinable**.

Theorem 9.1.6 Let D be a non-empty subset of $\mathbb{R}^d$, let $f : D \to \mathbb{R}$, and let (f_n) be a sequence of continuous functions from D to $\mathbb{R}$.
Suppose that the functions f_n converge uniformly on D to f.
Then f must also be continuous.

From now on **you may quote this theorem as standard**.

It may be summarized as follows:

Uniform limits of sequences of continuous functions are always continuous.

This theorem is one of the main reasons why uniform convergence is so important.

It sometimes gives you a quick way to see that certain pointwise convergent sequences do not converge uniformly.

If all the f_n are continuous but f isn't then your sequence cannot converge uniformly! (e.g. Example 1)

However, this trick does not always work.

Example 3) shows that a pointwise but non-uniform limit can sometimes be continuous.

We conclude this section with some additional standard facts which can sometimes help to establish that uniform convergence fails.

The proofs of these are an **exercise**.

Proposition 9.1.7 Let D be a non-empty subset of $\mathbb{R}^d$ and suppose that f is a **bounded** function from D to $\mathbb{R}$, i.e., $f(D)$ is a bounded subset of $\mathbb{R}$.

(i) Let $\varepsilon > 0$ and suppose that $g \in \bar{\boldsymbol{B}}_\varepsilon(f)$. Then g is also bounded on D.

(ii) Let (f_n) be a sequence of functions from D to $\mathbb{R}$. Suppose that all of the functions f_n are unbounded on D. Then (f_n) can not converge uniformly on D to f.

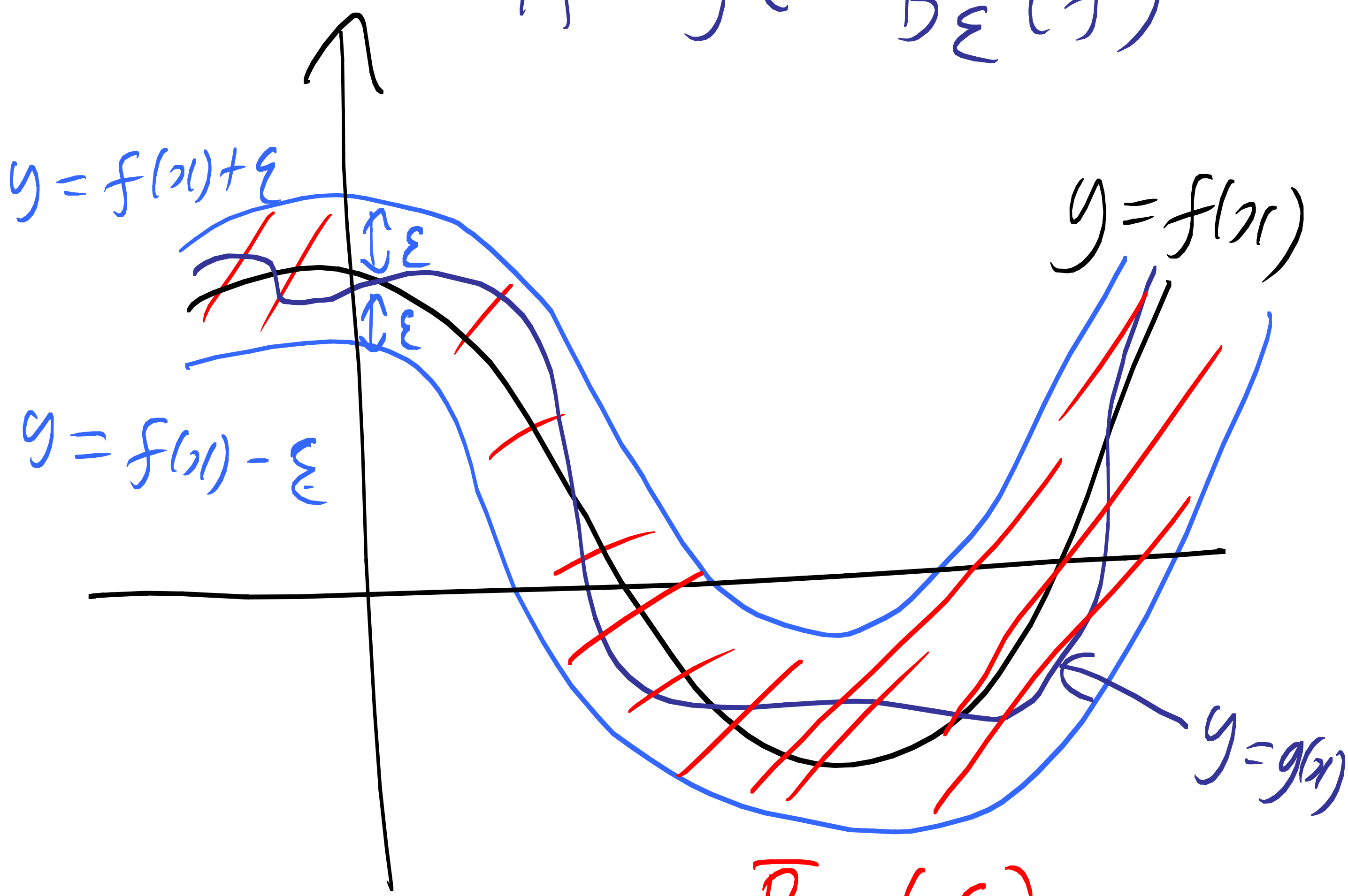
Another way to state this last result is:

$$\boxed{\text{It is impossible for a sequence of UNBOUNDED, real-valued functions from } D \text{ to } \mathbb{R} \text{ to converge uniformly on } D \text{ to a BOUNDED real-valued function.}}$$

Since $g \in \bar{\boldsymbol{B}}_\varepsilon(f) \Leftrightarrow f \in \bar{\boldsymbol{B}}_\varepsilon(g)$, a similar proof shows the following:

It is impossible for a sequence of BOUNDED functions from D to $\mathbb{R}$ to converge uniformly on D to an UNBOUNDED real-valued function.

If $g \in \overline{B}_\varepsilon(f)$



$\overline{B}_\varepsilon(f)$.

$f(x) - \varepsilon \le g(x) \le f(x) + \varepsilon$

for all x in the domain.

So if $-M \leq f(x) \leq M$ for all x in the domain, then

$$-M-\varepsilon \leq g(x) \leq M+\varepsilon$$

for all x in the domain.